

Matrix representation of the time operator

Carl M. Bender^{a*} and Mariagiovanna Gianfreda^{b†}

^a*Department of Physics, Kings College London, Strand, London WC2R 1LS, UK[‡]*

^b*Dipartimento di Fisica, Università del Salento and I.N.F.N. Sezione di Lecce, Via Arnesano, I-73100 Lecce, Italy*

(Dated: January 19, 2012)

In quantum mechanics the time operator Θ satisfies the commutation relation $[\Theta, H] = i$, and thus it may be thought of as being canonically conjugate to the Hamiltonian H . The time operator associated with a given Hamiltonian H is not unique because one can replace Θ by $\Theta + \Theta_{\text{hom}}$, where Θ_{hom} satisfies the homogeneous condition $[\Theta_{\text{hom}}, H] = 0$. To study this nonuniqueness the matrix elements of Θ for the harmonic-oscillator Hamiltonian are calculated in the eigenstate basis. This calculation requires the summation of divergent series, and the summation is accomplished by using zeta-summation techniques. It is shown that by including appropriate homogeneous contributions, the matrix elements of Θ simplify dramatically. However, it is still not clear whether there is an optimally simple representation of the time operator.

PACS numbers: 03.65.-w, 03.65.Fd, 11.30.-j, 11.30.Er

I. NONUNIQUENESS OF THE TIME OPERATOR FOR THE QUANTUM HARMONIC OSCILLATOR

The *time operator* Θ in quantum mechanics satisfies the commutation relation

$$[\Theta, H] = i, \quad (1)$$

where H is the Hamiltonian [1, 2]. Even though time is not a dynamical variable, one may regard Θ as being a time operator because if one allows a normalized state $|A\rangle$ at $t = 0$ to evolve to the state $|B\rangle = e^{-iHt}|A\rangle$ at time t , then as a consequence of (1),

$$\langle B|\Theta|B\rangle - \langle A|\Theta|A\rangle = t. \quad (2)$$

It is not known how to solve (1) for the time operator Θ for an arbitrary Hamiltonian H . Even for the comparatively simple case of the quantum-harmonic-oscillator Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 \quad (3)$$

the calculation of Θ is difficult because it involves singular operators such as $1/p$ and $1/q$. The time operator for the quantum harmonic oscillator was studied in detail at a formal level in Ref. [3]. The approach taken in [3] was to express Θ as a series of the form

$$\Theta = \sum_{m,n} a_{m,n} T_{m,n}, \quad (4)$$

where $a_{m,n}$ are numerical coefficients and $T_{m,n}$ (for integer m and n) are a set of Hermitian basis operators.

For $m, n \geq 0$ the operator $T_{m,n}$ is defined as a totally symmetric average over all possible orderings of m factors of p and n factors of q . For example,

$$\begin{aligned} T_{0,0} &= 1, \\ T_{1,0} &= p, \\ T_{1,1} &= \frac{1}{2}(pq + qp), \\ T_{1,2} &= \frac{1}{3}(pqq + qpq + qqp), \\ T_{2,2} &= \frac{1}{6}(ppqq + qqpp + pqqp + qppq + qpqp + pqpq), \end{aligned}$$

and so on. The operator $T_{m,n}$ is the quantum-mechanical generalization of the classical product $p^m q^n$.

It is easy to evaluate commutators and anticommutators of the operators $T_{m,n}$. For example, as shown in Ref. [4], the operators $T_{m,n}$ obey simple commutation and anticommutation relations:

$$\begin{aligned} [p, T_{m,n}] &= -inT_{m,n-1}, \\ [q, T_{m,n}] &= imT_{m-1,n}, \\ \{p, T_{m,n}\} &= 2T_{m+1,n}, \\ \{q, T_{m,n}\} &= 2T_{m,n+1}. \end{aligned} \quad (5)$$

By combining and iterating the results in (5), one can establish additional useful commutation relations for $T_{m,n}$:

$$\begin{aligned} [p^2, T_{m,n}] &= -2inT_{m+1,n-1}, \\ [q^2, T_{m,n}] &= 2imT_{m-1,n+1}. \end{aligned} \quad (6)$$

The totally symmetric operators $T_{m,n}$ can be re-expressed in Weyl-ordered form [4]:

$$\begin{aligned} T_{m,n} &= \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} p^k q^n p^{m-k} \\ &= \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} q^k p^m q^{n-k}, \end{aligned} \quad (7)$$

where $m, n = 0, 1, 2, 3, \dots$. The proof that $T_{m,n}$ equals the binomial-summation Weyl-ordered forms above is in-

[‡]Permanent address: Department of Physics, Washington University, St. Louis, MO 63130, USA.

*Electronic address: cmb@wustl.edu

†Electronic address: Maria.Gianfreda@le.infn.it

ductive and requires repeated use of the Heisenberg algebraic property that $[q, p] = i$. The advantage of introducing the Weyl-ordered form of $T_{m,n}$ is that it allows one to extend the totally symmetric operators $T_{m,n}$ either to negative values of n by using the first of these formulas or to negative values of m by using the second of these formulas. The commutation and anticommutation relations in (5) and (6) remain valid when m is negative or when n is negative.

In Ref. [3] it was shown that for the harmonic-oscillator Hamiltonian (3) the commutation relations in (6) may be used to obtain an exactly solvable recursion relation for the coefficients $a_{m,n}$ in (4). The exact solution for Θ has the series representation

$$\Theta_{\min} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} T_{2k+1, -2k-1}. \quad (8)$$

In Ref. [3] the solution in (8) was called the *minimal* solution because of its relative simplicity.

We emphasize here that Θ_{\min} is merely a *particular* solution to the inhomogeneous linear equation (1). To find other solutions to (1), we combine Θ_{\min} with solutions to the associated homogeneous equation

$$[H, \Theta_{\text{hom}}] = 0. \quad (9)$$

The subscript emphasizes that Θ_{hom} is a solution to the *homogeneous* equation (9).

In Ref. [3] it was noted that the solution to (1) is not unique and it was claimed that any two operators satisfying this commutation relation must differ by a function of the Hamiltonian H because this difference commutes with H . (We will see in Sec. II that this claim was actually not correct.)

One can interpret (8) by going to the classical limit $T_{m,n} \rightarrow p^m q^n$. In this limit the series (8) reduces to

$$\Theta_{\text{classical}} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} p^{2k+1} q^{-2k-1} = \arctan \frac{p}{q}. \quad (10)$$

Evidently, $\Theta_{\text{classical}}$ is conjugate to the harmonic-oscillator Hamiltonian in the sense of classical action-angle variables.

One can also interpret (8) at the quantum level by using the simple identity [5, 6]

$$T_{-n,n} = \frac{1}{2} \left(q \frac{1}{p} \right)^n + \frac{1}{2} \left(\frac{1}{q} p \right)^n. \quad (11)$$

This formula allows us to sum the series in (8) at the operator level (where the ordering of operators is crucial):

$$\begin{aligned} \Theta_{\min} &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4k+2} (p^{2k+1} q^{-2k-1} + q^{-2k-1} p^{2k+1}) \\ &= \frac{1}{2} \arctan \left(p \frac{1}{q} \right) + \frac{1}{2} \arctan \left(\frac{1}{q} p \right). \end{aligned} \quad (12)$$

It is important to observe that the formal summation in (12) produces a *bounded* operator even though the individual terms in the summation are unbounded operators. (The same effect can be seen in the expansion $e^{iq} = \sum_{n=0}^{\infty} i^n q^n / n!$.) The fact that infinite sums of ill-behaved objects can be well behaved will be used throughout this paper.

Observe that if the Hamiltonian H is symmetric under parity reflection $[H, \mathcal{P}] = 0$, then as a consequence of (1) the time operator Θ will also be symmetric under parity reflection. Also, if H is symmetric under time reversal $[H, \mathcal{T}] = 0$, then Θ will be *antisymmetric* under \mathcal{T} because the time-reversal operator changes the sign of i . (Note that if H is non-Hermitian but \mathcal{PT} symmetric, then Θ will be \mathcal{PT} antisymmetric [7].) The Hamiltonian for the quantum harmonic oscillator is both \mathcal{P} and \mathcal{T} symmetric because under parity reflection $p \rightarrow -p$ and $q \rightarrow -q$ and under time reversal $p \rightarrow -p$, $q \rightarrow q$, and $i \rightarrow -i$. Thus, the minimal solution in (8) and (12) exhibits the expected \mathcal{P} symmetry and \mathcal{T} antisymmetry.

The question addressed in this paper is similar to that addressed in Ref. [5, 6] in which the uniqueness of the \mathcal{C} operator in \mathcal{PT} quantum theory was examined [7]. Because the time operator is not unique, our objective here is to determine whether there is a solution to (1) that is optimal in some sense. Our approach will be to calculate matrix elements of the time operator Θ and to ask whether there is a choice of homogeneous solution Θ_{hom} which, when added to Θ_{\min} , gives particularly simple results.

In Sec. II we construct series representations for the solutions to the homogeneous commutator equation (9). Then in Sec. III we show how to use zeta-function regulation to express matrix elements of the singular operators in these series representations as polynomials. Next, in Sec. IV we discuss the detailed mathematical structure of the polynomials obtained in Sec. III. Finally, in Sec. V we give some concluding remarks. In the Appendix we explain the details of the zeta-function regulation techniques used in Sec. III.

II. HOMOGENEOUS SOLUTIONS TO EQ. (1) FOR THE HARMONIC OSCILLATOR

In this section we show how to obtain homogeneous (nonminimal) solutions to the commutator equation (1) for the harmonic-oscillator Hamiltonian (3). We will seek solutions of the general form (4). These solutions can be expressed as a single (not a double) sum on the index k and can be labeled by an integer parameter γ .

A. Homogeneous solutions odd in p and odd in q

In this paper we are only interested in homogeneous solutions that are odd in p and odd in q . The fact that H in (3) is symmetric under space reflection implies that

Θ cannot be odd in p and even in q , or even in p and odd in q . Furthermore, since H is symmetric under time reversal, Θ must be either real and odd in p and odd in q , or else imaginary and even in p and even in q . In the latter case, however, Θ will just turn out to be a function of H , and therefore the matrix elements of Θ in the harmonic-oscillator-eigenfunction basis will be diagonal. The point of this paper is to use the homogeneous solutions to simplify the matrix elements of the time operator, and therefore even-even solutions will not be of any value. Thus, we focus here on odd-odd homogeneous solutions. Solutions having this symmetry property can be written in the general form

$$\Theta_{\text{hom}}^{(\gamma)} = \sum_{k=-\infty}^{\infty} a_k^{(\gamma)} T_{-2k+2\gamma-1, 2k+1}. \quad (13)$$

Substituting (13) into (9) gives the recursion relation

$$(2k+1-2\gamma)a_k^{(\gamma)} + (2k+3)a_{k+1}^{(\gamma)} = 0 \quad (14)$$

for $-\infty < k < \infty$. Thus,

$$a_k^{(\gamma)} = (-1)^k a_0^{(\gamma)} \frac{\sqrt{\pi} \Gamma(k-\gamma+1/2)}{2\Gamma(1/2-\gamma)\Gamma(3/2+k)}, \quad (15)$$

where $a_0^{(\gamma)}$ is an arbitrary constant for each value of γ .

The totally symmetric operators $T_{m,n}$ that contribute to $\Theta_{\text{hom}}^{(\gamma)}$ are shown in Fig. 1 as dots. For each γ these dots lie on a diagonal line. The dots that contribute to $\Theta_{\text{min}}^{(\gamma)}$ lie on a bold diagonal line.

To evaluate the sum in (13) for positive integers γ , we first use the symmetry property $a_k^{(\gamma)} = a_{-k-1+\gamma}^{(\gamma)}$ to split the sum into two sums. Then, we use the anticommutator relations in (5) to obtain

$$\begin{aligned} \Theta_{\text{hom}}^{(\gamma)} = & \frac{1}{4^\gamma} \sum_{k=0}^{\infty} a_k^{(\gamma)} \left[\{q, \dots \{q, \{q, T_{2k+1, -2k-1}\}\} \dots\}_{(2\gamma) \text{ times}} \right. \\ & \left. + \{p, \dots \{p, \{p, T_{-2k-1, 2k+1}\}\} \dots\}_{(2\gamma) \text{ times}} \right] + \sum_{k=0}^{\gamma-1} a_k^{(\gamma)} T_{2k+1, -2k-1+2\gamma}. \end{aligned} \quad (16)$$

Finally, we use the identity (11) to simplify the totally symmetric operators of the form $T_{-m,m}$. (This identity is valid for both positive- and negative-integer m .) This gives the result

$$\begin{aligned} \Theta_{\text{hom}}^{(\gamma)} = & \frac{1}{2^{2\gamma+1}} \sum_{k=0}^{\infty} a_k^{(\gamma)} \left[\left\{ q, \dots \left\{ q, \left\{ q, \left(\frac{p}{q} \right)^{2k+1} + \left(\frac{1}{q} \right)^{2k+1} \right\} \dots \right\} \right\}_{(2\gamma) \text{ times}} \right. \\ & \left. + \left\{ p, \dots \left\{ p, \left\{ p, \left(\frac{1}{p} \right)^{2k+1} + \left(\frac{q}{p} \right)^{2k+1} \right\} \dots \right\} \right\}_{(2\gamma) \text{ times}} \right] + \sum_{k=0}^{\gamma-1} a_k^{(\gamma)} T_{2k+1, -2k-1+2\gamma}, \end{aligned} \quad (17)$$

which we emphasize is only valid for positive-integer γ . It is possible to sum each of the four infinite series in (17) in terms of a hypergeometric function

$$\sum_{k=0}^{\infty} x^{2k} a_k^{(\gamma)} = a_0^{(\gamma)} {}_2F_1 \left(\frac{1}{2} - \gamma, 1, \frac{3}{2}, -x^2 \right), \quad (18)$$

but there is no simple way to sum the finite series in (17).

The coefficients in (15) are also valid for negative-integer γ , but now because γ is negative, we use the commutator relations in (5) (instead of the anticommutator relations) to obtain the general formula

$$\begin{aligned} T_{-m-2\gamma, m} = & \frac{(-1)^\gamma (m-1)!}{(m+2\gamma-1)!} \\ & \times [\dots [T_{-m, m}, q], q] \dots, q]_{(2\gamma) \text{ times}}, \end{aligned} \quad (19)$$

and then we proceed as above.

It is easy to see that $\Theta_{\text{hom}}^{(\gamma)}$ is *not* a function of H only. This is because the operator $\Theta_{\text{hom}}^{(\gamma)}$ changes sign under time reversal \mathcal{T} while H does not change sign. Furthermore, if $\Theta_{\text{hom}}^{(\gamma)}$ were a function of H only, then its matrix elements in the harmonic-oscillator-eigenstate basis would be diagonal. We will see in Sec. III that the matrix elements are not diagonal.

B. Homogeneous solutions odd in p and even in q

In this paper we are only interested in homogeneous solutions that are odd in both p and q , but we point out

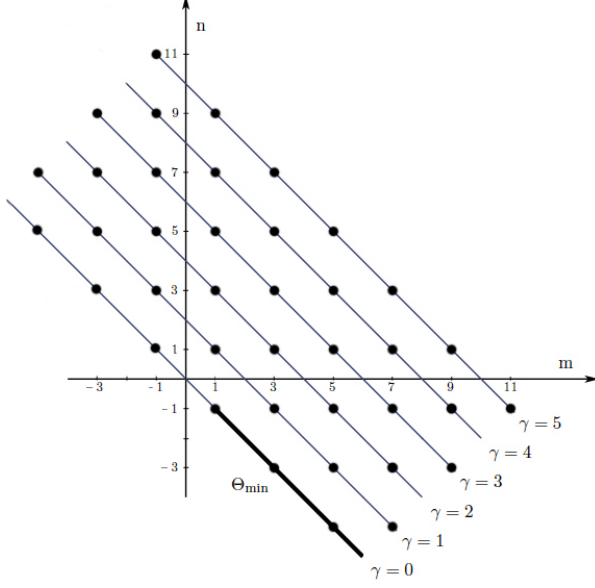


FIG. 1: Schematic representation indicating the totally symmetric operators $T_{m,n}$ that contribute to the sum in (8) for Θ_{\min} and to the sum in (13) for $\Theta_{\text{hom}}^{(\gamma)}$. Each nonzero term in these sums is indicated by a dot. For each value of γ these dots lie on diagonal lines. The dots associated with Θ_{\min} lie on a bold diagonal line.

that there are also solutions to (9) having other symmetry properties. For example, solutions that are odd in p and even in q have the general form

$$\Theta_{\text{hom}}^{(\gamma)} = \sum_k a_k^{(\gamma)} T_{2\gamma+1-2k, 2k}, \quad (20)$$

where the parameter $\gamma = 0, \pm 1, \pm 2, \dots$.

Substituting (20) into (9), we obtain the following two-term recursion relation for the coefficients $a_k^{(\gamma)}$:

$$a_{k+1}^{(\gamma)}(k+1) - (\gamma - k + 1/2)a_k^{(\gamma)} = 0 \quad (k = 0, 1, 2, \dots). \quad (21)$$

Unlike the recursion relation (14), this recursion relation is self-terminating; that is, if we choose $a_{-1}^{(\gamma)} = 0$, then $a_k^{(\gamma)} = 0$ for $k < 0$, $a_0^{(\gamma)}$ is arbitrary, and $a_k^{(\gamma)} \neq 0$ for $k > 0$. In terms of $a_0^{(\gamma)}$ the solution to this recursion relation is

$$a_k^{(\gamma)} = a_0^{(\gamma)} (-1)^k \frac{\Gamma(k - \gamma - 1/2)}{k! \Gamma(-\gamma - 1/2)} \quad (k = 0, 1, 2, \dots). \quad (22)$$

The series (20) with coefficients (22) can be summed as a binomial expansion:

$$\sum_{k=0}^{\infty} a_k^{(\gamma)} x^{2k} = a_0^{(\gamma)} (1 + x^2)^{\gamma+1/2}. \quad (23)$$

Thus, for $\gamma \geq 0$ the odd-even one-parameter family of solutions to the homogeneous equation (9) is

$$\Theta_{\text{hom}}^{(\gamma)} = \frac{a_0^{(\gamma)}}{2^{2\gamma+2}} \left\{ \dots \left\{ \left\{ \left(1 + q \frac{1}{p} q \frac{1}{p} \right)^{\gamma+1/2} + \left(1 + \frac{1}{p} q \frac{1}{p} q \right)^{\gamma+1/2}, p \right\}, p \right\} \dots, p \right\}_{(2\gamma+1) \text{ times}}. \quad (24)$$

For example, for $\gamma = 0$

$$\Theta_{\text{hom}}^{(0)} = \frac{1}{4} a_0^{(0)} \left(\sqrt{1 + q \frac{1}{p} q \frac{1}{p}} p + p \sqrt{1 + q \frac{1}{p} q \frac{1}{p}} + \sqrt{1 + \frac{1}{p} q \frac{1}{p} q} p + p \sqrt{1 + \frac{1}{p} q \frac{1}{p} q} \right). \quad (25)$$

For $\gamma < 0$ the result is similar, but it involves commutators. For $\gamma = -1$, we get

$$\Theta_{\text{hom}}^{(-1)} = \frac{a_0^{(-1)}}{2i} \left[p, \text{arcsinh} \left(q \frac{1}{p} \right) + \text{arcsinh} \left(\frac{1}{p} q \right) \right]. \quad (26)$$

We can see again from (24)–(26) that, as we stated in Sec. I, while $\Theta_{\text{hom}}^{(\gamma)}$ commutes with the Hamiltonian H in (3), it is not a function of H . We verify this by observing that $\Theta_{\text{hom}}^{(\gamma)}$ is odd under parity reflection \mathcal{P} while H is even. Moreover, in the classical limit, where p and q

commute, (25) simplifies to

$$\Theta_{\text{hom}}^{(0)} = a_0^{(0)} \text{sgn}(p) \sqrt{2H}. \quad (27)$$

The solutions in (24)–(27) are solutions to the commutator equation (9), but they are physically unacceptable because they violate time-reversal symmetry. Thus, they must be rejected. However, the homogeneous solutions in (13), whose numerical coefficients are given in (15), are physically acceptable because they exhibit the correct behavior under time and space reflection. This raises the question of which solution for the time operator Θ is optimal and in what sense it is optimal. We address this issue in the next section.

III. MATRIX ELEMENTS OF THE TIME OPERATOR

We showed in Sec. II that there are many possible choices for the time operator Θ because there are an infinite number of one-parameter families of homogeneous solutions to (9). One criterion for deciding on the optimal choice for the time operator (if there is an optimal choice) is mathematical simplicity. To investigate this criterion, in this section we calculate the matrix elements of the time operator in a basis consisting of harmonic-oscillator eigenfunctions. We begin by calculating the matrix elements of the minimal solution (8). Then, we ask whether these matrix elements simplify when combined with homogeneous solutions of the type in (13).

In coordinate space the n th normalized eigenfunction of the harmonic oscillator is

$$\psi_n(q) = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} e^{-q^2/2} H_n(q). \quad (28)$$

If we try to calculate the matrix elements of Θ_{\min} or Θ_{hom} in this basis, we run into difficulties because each term in the series representation for these operators is singular (because powers of $1/p$ or $1/q$ appear.) However, in this section we propose a method for circumventing these difficulties and thus for obtaining a finite result for the matrix elements. Specifically, we argue that while individual terms in the series representation for the time operator are singular, the sum of the series is a bounded operator [see, for example, (12)]. We then show that the (m, n) matrix element of a given singular operator in the sum can be made finite depending on the evenness or oddness of m and n and also depending on whether the singular operator is taken to act to the left or to the right.

A. Matrix representation of the minimal solution

To calculate the (m, n) matrix element of Θ_{\min} , we must calculate the matrix element of each individual term $T_{2k+1, -2k-1}$ in the series (8). The parity of this operator is even, so only the nonzero matrix elements correspond to values of m and n that are both odd or both even. Referring to (11), we can see that in coordinate space the (m, n) matrix element of $T_{2k+1, -2k-1}$ has the form

$$\begin{aligned} & \frac{1}{2} (-i)^{2k+1} \int_{-\infty}^{\infty} dq \psi_m(q) \left(\frac{1}{q} \partial_q \right)^{2k+1} \psi_n(q) \\ & + \frac{1}{2} (-i)^{2k+1} \int_{-\infty}^{\infty} \psi_m(q) \left(\partial_q \frac{1}{q} \right)^{2k+1} \psi_n(q). \end{aligned} \quad (29)$$

Note that the first integral exists if n is even and the second integral exists if n is odd. When n is even, we reinterpret the differential operator in the second integral as operating to the left, and when n is odd we reinterpret the differential operator in the first integral as operating to the left. (Note that when the operator acts to the left, there is an additional minus sign because $2k+1$ is odd.)

The only nonzero matrix elements of Θ_{\min} are

$$\begin{aligned} & \langle 2n-2j | \Theta_{\min} | 2n \rangle \\ & = -\frac{i}{2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \langle 2n-2j | \left(\frac{1}{q} \partial_q \right)^{2k+1} | 2n \rangle, \\ & \langle 2n-2j-1 | \Theta_{\min} | 2n-1 \rangle \\ & = -\frac{i}{2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \langle 2n-2j-1 | \left(\partial_q \frac{1}{q} \right)^{2k+1} | 2n-1 \rangle, \\ & \langle 2n | \Theta_{\min} | 2n-2j \rangle = -\langle 2n-2j | \Theta_{\min} | 2n \rangle, \\ & \langle 2n-1 | \Theta_{\min} | 2n-2j-1 \rangle \\ & = -\langle 2n-2j-1 | \Theta_{\min} | 2n-1 \rangle, \end{aligned} \quad (30)$$

where $j = 1, 2, 3, \dots$ and $n \geq j$. Note that the diagonal matrix elements vanish: $\langle m | \Theta_{\min} | m \rangle = 0$. In Fig. 2 the nonzero matrix elements are shown as dots and the vanishing matrix elements are shown as zeros.

The calculation of the matrix elements in (30) is lengthy, and we have relegated the discussion to the Appendix. Here, we merely report the results of the calculation. All the even-even and odd-odd matrix elements of the minimal solution (8) can be expressed compactly as

$$\begin{aligned} \langle 2n-2j | \Theta_{\min} | 2n \rangle & = i(-1)^j 2^{j/2-1} (j-1)! \\ & \quad \times \sqrt{\frac{n!(2n-2j-1)!!}{(n-j)!(2n-1)!!}} F(j), \\ \langle 2n+1-2j | \Theta_{\min} | 2n+1 \rangle & = i(-1)^{j+1} 2^{j/2-1} (j-1)! \\ & \quad \times \sqrt{\frac{n!(2n-2j+1)!!}{(n-j)!(2n+1)!!}} F(j), \end{aligned} \quad (31)$$

where

$$F(j) = \sum_{\rho=1}^j \frac{2^\rho}{\Gamma(\rho)\Gamma(1-\rho+j)} Z(\rho), \quad (32)$$

$$Z(\rho) = \frac{\delta_{\rho,1}}{2} - \sum_{i=1}^{\rho} \frac{S_{\rho,i}^1}{\rho!} \sum_{r=1}^{\lfloor i/2+1 \rfloor} \binom{i-1}{2r-1} 2^{2r-2} \frac{\mathcal{B}_{2r}}{r}, \quad (33)$$

\mathcal{B}_{2r} are Bernoulli numbers, and $S_{\rho,i}^1$ are Stirling numbers of the first kind. Table I gives the first eight values of the functions $Z(\rho)$ and $F(j)$.

The explicit values of some matrix elements are then

$$\begin{aligned} \langle 2n-1 | \Theta_{\min} | 2n+1 \rangle & = \frac{i}{2} \sqrt{\frac{2n}{(2n+1)}}, \\ \langle 2n-2 | \Theta_{\min} | 2n \rangle & = -\frac{i}{2} \sqrt{\frac{2n}{(2n-1)}}, \\ \langle 2n-3 | \Theta_{\min} | 2n+1 \rangle & = -\frac{2i}{3} \sqrt{\frac{n(n-1)}{(2n+1)(2n-1)}}, \\ \langle 2n-4 | \Theta_{\min} | 2n \rangle & = \frac{2i}{3} \sqrt{\frac{n(n-1)}{(2n-1)(2n-3)}}, \end{aligned}$$

$$\Theta_{\min} = \begin{pmatrix} 0 & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & \dots \\ 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \dots \\ \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & \dots \\ 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \dots \\ \cdot & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & \cdot & \dots \\ 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & \dots \\ \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot & \dots \\ 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & \dots \\ \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \cdot & \dots \\ 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \dots \\ \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & 0 & \dots \\ 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 & \dots \\ \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

FIG. 2: Schematic view of the matrix elements of the minimal time operator Θ_{\min} . The nonvanishing matrix elements are shown as dots.

j	$F(j)$	ρ	$Z(\rho)$
1	1	1	1/2
2	2/3	2	-1/12
3	5/18	3	1/36
4	11/135	4	-1/90
5	101/5400	5	1/200
6	593/170100	6	-113/45360
7	2623/4762800	7	29/21168
8	383/5103000	8	-47/56700

TABLE I: First eight values of the functions $F(j)$ in (32) and $Z(\rho)$ in (33).

$$\begin{aligned} \langle 2n-5|\Theta_{\min}|2n+1\rangle &= \frac{5i}{9} \sqrt{\frac{2n(n-1)(n-2)}{(2n+1)(2n-1)(2n-3)}}, \\ \langle 2n-6|\Theta_{\min}|2n\rangle &= -\frac{5i}{9} \sqrt{\frac{2n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}}, \\ \langle 2n-7|\Theta_{\min}|2n+1\rangle &= -\frac{44i}{45} \sqrt{\frac{n(n-1)(n-2)(n-3)}{(2n+1)(2n-1)(2n-3)(2n-5)}}, \\ \langle 2n-8|\Theta_{\min}|2n\rangle &= \frac{44i}{45} \sqrt{\frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)(2n-5)(2n-7)}}. \end{aligned}$$

B. Matrix representations of the homogeneous solutions

The question is whether we can simplify the result in (31) by including the homogeneous solutions in (13). Let us first examine the contribution from the homogeneous solution corresponding to $\gamma = 0$. We find that the nonzero matrix elements for $\Theta_{\text{hom}}^{(0)}$ lie on *alternating* diagonals in Fig. 2; that is, the matrix elements are nonzero on the first dotted diagonals immediately above and immediately below the main diagonal. On the next dotted diagonals immediately above and below these dotted diagonals, the matrix elements vanish; on the next dotted diagonals immediately above and below these dotted diagonals, the matrix elements do not vanish; and so on. Specifically, we find that

$$\langle n-4j+2|\Theta_{\text{hom}}^{(0)}|n\rangle = -2a_0^{(0)}\langle n-4j+2|\Theta_{\min}|n\rangle \quad (34)$$

for $j = 1, 2, 3, \dots$ and $n \geq 4j-2$. On the diagonals on which the matrix elements of $\Theta_{\text{hom}}^{(0)}$ do not vanish, each matrix element is a multiple of the matrix elements of Θ_{\min} . Hence, we can choose $a_0^{(0)} = 1/2$ in (15) and cancel half of the nonzero matrix elements of Θ_{\min} . Thus, including this homogeneous solution causes a dramatic simplification of the matrix elements of the time operator!

Next, we consider the matrix elements of $\Theta_{\text{hom}}^{(\gamma)}$ for $\gamma \geq 1$. Here are some useful properties of the matrix representation of the operators $T_{m,n}$:

$$\begin{aligned} \langle n-4j|T_{m,k}|n\rangle &= -\langle n-4j|T_{k,m}|n\rangle, \\ \langle n-4j|T_{k,k}|n\rangle &= 0, \end{aligned} \quad (35)$$

for $n \geq 4j$ and for $k, m \geq 1$. For fixed γ , only the columns with $j \leq \lfloor \frac{\gamma+1}{2} \rfloor$ are not empty. This means that the nonzero matrix elements of the operators $\Theta_{\text{hom}}^{(\gamma)}$ for $\gamma = 1$ and $\gamma = 2$ lie on just two diagonals above and two diagonals below the main diagonal; the nonzero matrix elements for $\gamma = 3$ and $\gamma = 4$ lie on two and six diagonals above and below the main diagonal; the nonzero matrix elements for $\gamma = 5$ and $\gamma = 6$ lie on two, six, and ten diagonals above and below the main diagonal; and so on. (The diagonals containing nonzero matrix elements are indicated in Fig. 3 by dots.) To summarize,

FIG. 3: Schematic representation of the matrix elements for homogeneous solutions $\Theta_{\text{hom}}^\gamma$ for $\gamma = 1, 2, \dots, 6$. The dots indicate the nonzero matrix elements. For $\gamma = 1$ and $\gamma = 2$ there are only two diagonals where the matrix elements are nonzero. When $\gamma = 3, 4$ there are four diagonals of nonzero matrix elements, when $\gamma = 5, 6$ there are six diagonals, and so on.

$$\langle n - 4j + 2 | \Theta_{\text{hom}}^{(2j+\sigma-1)} | n \rangle = \sum_{k=0}^{2j+\sigma-2} a_k^{(2j+\sigma-1)} \langle n - 4j + 2 | T_{2k+1, 2(2j+\sigma-k-1)-1} | n \rangle, \quad (36)$$

where we have substituted $\gamma = 2j + \sigma - 1$ with $\sigma = 0, 1, 2, \dots$. The coefficients $a_k^{(\gamma)}$ are given in (15).

The matrix elements of the operators $T_{m,n}$ are associated with interesting classes of orthogonal polynomials. In particular, we are now interested in the case m, n odd. The j th class of orthogonal polynomials $\{\mathcal{P}_k^{(j)}(n)\}_{k=0,1,\dots}$ is defined by the equation

$$\langle n - 2j | T_{2k+1, 2\ell+1} | n \rangle = -i \sqrt{\frac{n!}{(n-2j)!}} \mathcal{C}_{\ell,k}^{(j)} \mathcal{P}_{\ell+k-j+1}^{(j)}(n) \quad (k, \ell \geq 0, j \geq 1, n \geq 2j) \quad (37)$$

where

$$\mathcal{C}_{\ell,k}^{(j)} = \frac{(-1)^k j(\ell+k)!(2k+2)!(2\ell+2)!}{2^{\ell+k+2}(\ell+1)!(\ell+k+j+1)!(\ell+k-j+1)!(k+1)!}. \quad (38)$$

The polynomials $\mathcal{P}_k^{(j)}(n)$ ($j = 1, 2, \dots$) have degree k and argument n and are discussed in detail in the next section. Substituting (37) with $j \rightarrow 2j - 1$ and $\ell = 2j + \sigma - k - 2$ into (36), we obtain the following expression for the nonvanishing diagonals of elements in the matrix representation of the homogeneous solutions $\Theta_{\text{hom}}^{(\gamma)}$:

$$\langle n - 4j + 2 | \Theta_{\text{hom}}^{(2j-1+\sigma)} | n \rangle = -i a_0^{(\sigma+2j-1)} \frac{(2j-1)(2\sigma+4j-2)!}{2^{2j+\sigma-1}(\sigma+4j-2)!\sigma!} \sqrt{\frac{n!}{(n-4j+2)!}} \mathcal{P}_\sigma^{(2j-1)}(n), \quad (39)$$

where $j = 1, 2, 3, \dots, n \geq 4j - 2$, and $\sigma = 0, 1, 2, \dots$

IV. POLYNOMIAL REPRESENTATIONS OF THE HOMOGENEOUS SOLUTIONS

In this section we describe the properties of the polynomials in (37). The polynomials $\mathcal{P}_k^{(j)}(x)$ are orthogonal and satisfy the three-term recursion relation

$$\begin{aligned} \mathcal{P}_{k+1}^{(j)}(x) &= \frac{1}{2}(2x - 2j + 1)\mathcal{P}_k^{(j)}(x) \\ &+ \frac{k}{4}(k + 2j)\mathcal{P}_{k-1}^{(j)}(x), \end{aligned} \quad (40)$$

where $\mathcal{P}_0^{(j)}(x) = 1$. The generating function

$$G^{(j)}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{P}_k^{(j)}(x) \quad (41)$$

for these polynomials is

$$G^{(j)}(t) = 2^{2j+1} \frac{(2+t)^{x-2j}}{(2-t)^{x+1}}. \quad (42)$$

From this generating function we identify these polynomials as Meixner polynomials of the first kind [9, 10],

whose definition and properties are elucidated in the next subsection. In this paper $j \in \mathbb{N}^+$, but if we set $j = 0$, we obtain the Hahn polynomials that were studied in Ref. [11].

The polynomials $\mathcal{P}_k^{(j)}(x)$ do not have parity. However, if we shift the argument by $x = iy + (2j - 1)/2$ and define $\mathcal{Q}_k^{(j)}(y) = (-i)^k \mathcal{P}_k^{(j)}[iy + (2j - 1)/2]$, then the new polynomials *do* have parity. Below we give the first six polynomials $\mathcal{Q}_k^{(j)}(y)$ for $j = 1$ through $j = 5$:

$$\begin{aligned} \mathcal{Q}_0^{(1)}(y) &= 1, & \mathcal{Q}_1^{(1)}(y) &= y, \\ \mathcal{Q}_2^{(1)}(y) &= y^2 - \frac{3}{4}, & \mathcal{Q}_3^{(1)}(y) &= y^3 - \frac{11}{4}y, \\ \mathcal{Q}_4^{(1)}(y) &= y^4 - \frac{13}{2}y^2 + \frac{45}{16}, \\ \mathcal{Q}_5^{(1)}(y) &= y^5 - \frac{25}{2}y^3 + \frac{309}{16}, \end{aligned} \quad (43)$$

$$\begin{aligned} \mathcal{Q}_0^{(2)}(y) &= 1, & \mathcal{Q}_1^{(2)}(y) &= y, \\ \mathcal{Q}_2^{(2)}(y) &= y^2 - \frac{5}{4}, & \mathcal{Q}_3^{(2)}(y) &= y^3 - \frac{17}{4}y, \\ \mathcal{Q}_4^{(2)}(y) &= y^4 - \frac{19}{2}y^2 + \frac{105}{16}, \\ \mathcal{Q}_5^{(2)}(y) &= y^5 - \frac{35}{2}y^3 + \frac{649}{16}y, \end{aligned} \quad (44)$$

$$\begin{aligned} \mathcal{Q}_0^{(3)}(y) &= 1, & \mathcal{Q}_1^{(3)}(y) &= y, \\ \mathcal{Q}_2^{(3)}(y) &= y^2 - \frac{7}{4}, & \mathcal{Q}_3^{(3)}(y) &= y^3 - \frac{23}{4}y, \\ \mathcal{Q}_4^{(3)}(y) &= y^4 - \frac{25}{2}y^2 + \frac{189}{16}, \\ \mathcal{Q}_5^{(3)}(y) &= y^5 - \frac{45}{2}y^3 + \frac{1109}{16}y, \end{aligned} \quad (45)$$

$$\begin{aligned} \mathcal{Q}_0^{(4)}(y) &= 1, & \mathcal{Q}_1^{(4)}(y) &= y, \\ \mathcal{Q}_2^{(4)}(y) &= y^2 - \frac{9}{4}, & \mathcal{Q}_3^{(4)}(y) &= y^3 - \frac{29}{4}y, \\ \mathcal{Q}_4^{(4)}(y) &= y^4 - \frac{31}{2}y^2 + \frac{297}{16}, \\ \mathcal{Q}_5^{(4)}(y) &= y^5 - \frac{55}{2}y^3 + \frac{1689}{16}y, \end{aligned} \quad (46)$$

$$\begin{aligned} \mathcal{Q}_0^{(5)}(y) &= 1, & \mathcal{Q}_1^{(5)}(y) &= y, \\ \mathcal{Q}_2^{(5)}(y) &= y^2 - \frac{11}{4}, & \mathcal{Q}_3^{(5)}(y) &= y^3 - \frac{35}{4}y, \\ \mathcal{Q}_4^{(5)}(y) &= y^4 - \frac{37}{2}y^2 + \frac{429}{16}, \\ \mathcal{Q}_5^{(5)}(y) &= y^5 - \frac{65}{2}y^3 + \frac{2389}{16}y, \end{aligned} \quad (47)$$

The polynomials $\mathcal{Q}_k^{(j)}(y)$ are orthogonal on $(-\infty, \infty)$ with respect to an even, positive-weight function $w^{(j)}(y)$:

$$\int_{-\infty}^{\infty} dy w^{(j)}(y) \mathcal{Q}_r^{(j)}(y) \mathcal{Q}_s^{(j)}(y) = 0 \quad (48)$$

for $r \neq s$. We use the notation

$$\mu_k^{(j)} = \int_{-\infty}^{\infty} dy w^{(j)}(y) y^{2k}, \quad (49)$$

to represent the moments of the weight functions $w^{(j)}(y)$. These moments are closely related to the Euler numbers E_n . In particular, for the case $j = 0$ (see Ref. [6])

$$\mu_k^{(0)} = |E_{2k}| 4^{-k}. \quad (50)$$

In Table II we list the values of the moments for j ranging from 1 through 6 and for k ranging from 0 through 6.

The moments defined in (49) and displayed in Table II have some remarkable mathematical properties. To list these properties we first express these moments as polynomials $\mathcal{R}_k(j) = 4^k \mu_k^{(j)} / (2j + 1)$ of degree k in the argument j . The first six of these polynomials are

$$\begin{aligned} \mathcal{R}_1(j) &= 1, \\ \mathcal{R}_2(j) &= 6j + 5, \\ \mathcal{R}_3(j) &= 60j^2 + 120j + 61, \\ \mathcal{R}_4(j) &= 840j^3 + 2940j^2 + 3486j + 1385, \\ \mathcal{R}_5(j) &= 15120j^4 + 80640j^3 + 163800j^2 \\ &\quad + 148800j + 50512, \\ \mathcal{R}_6(j) &= 332640j^5 + 2494800j^4 + 7595280j^3 \\ &\quad + 11641080j^2 + 8910726j + 2702765. \end{aligned} \quad (51)$$

We then observe that these new polynomials resemble the Bernoulli polynomials and the Euler polynomials in that we recover the Bernoulli numbers B_{2k} and the Euler numbers E_{2k} when we evaluate the polynomials at special values of j . As we saw in (50), we obtain the Euler number $|E_{2k}|$ when we evaluate $\mathcal{R}_k(j)$ at $j = 0$:

$$\mathcal{R}_k(0) = |E_{2k}|. \quad (52)$$

We obtain the Bernoulli numbers when we evaluate $\mathcal{R}_k(j)$ at $j = -1/2$:

$$\mathcal{R}_k(-1/2) = \frac{|B_{2k}|}{2k} (4^k - 1) 4^k. \quad (53)$$

There are many other values of j for which the polynomials $\mathcal{R}_k(j)$ give simple results:

$$\begin{aligned} \mathcal{R}_k(-3/2) &= (-4)^{k-1}, \\ \mathcal{R}_k(-1) &= (-1)^{k-1}, \\ \mathcal{R}_k(1/2) &= \frac{|B_{2k+2}|}{k+1} (4^{k+1} - 1) 4^k, \\ \mathcal{R}_k(-5/2) &= (-1)^{k-1} 2^{2k-3} (2^{2k-2} + 1), \\ \mathcal{R}_k(-2) &= \frac{(-1)^{k+1}}{4} (3^{2k-1} + 1). \end{aligned} \quad (54)$$

k	$4^k \mu_k^{(1)}$	$4^k \mu_k^{(2)}$	$4^k \mu_k^{(3)}$	$4^k \mu_k^{(4)}$	$4^k \mu_k^{(5)}$	$4^k \mu_k^{(6)}$
0	1	1	1	1	1	1
1	3	5	7	9	11	13
2	33	85	161	261	385	533
3	723	2705	6727	13509	23771	38233
4	25953	134185	426881	1045161	2170465	4024553
5	1376643	9451805	37611847	110683809	268614731	570100453
6	101031873	892060285	4355312801	15209937261	42750877345	103164046973

TABLE II: Numerical values of the moments $\mu_k^{(j)}$ defined in (49).

A. Connection between $\mathcal{P}_k^{(j)}(x)$ and Meixner/Meixner-Pollaczek polynomials.

The polynomials $\mathcal{P}_k^{(j)}(x)$ defined in (37) satisfy the three-term recursion (40) and are related to the Meixner polynomials:

$$\mathcal{P}_k^{(j)}(x) = 2^{-k} \mathcal{M}_k(x - 2j, 2j + 1, -1), \quad (55)$$

where the Meixner polynomials $\mathcal{M}_n(x, b, c)$ belong to the classical orthogonal polynomials of a discrete variable and may be defined in terms of hypergeometric functions:

$$\mathcal{M}_n(x, b, c) = \frac{\Gamma(b + k)}{\Gamma(b)} {}_2F_1(-k, -x; b; 1 - c^{-1}). \quad (56)$$

For real values of the parameters $b > 0$ and $0 < c < 1$ they satisfy a discrete orthogonality relation with weight function

$$w(x) = \frac{\Gamma(b + x)c^x}{\Gamma(b)x!}, \quad (57)$$

whose moments are

$$\tilde{\mu}_k^{(b,c)} = \sum_{n=0}^{\infty} \frac{\Gamma(b + n) c^n n^k}{\Gamma(b) n!}. \quad (58)$$

The problem with evaluating the moments associated with the polynomials $\mathcal{P}_k^{(j)}(x)$, that is, the moments of the Meixner polynomials with $c = -1$, can be solved by performing a zeta-function regularization of the divergent series in (58):

$$\tilde{\mu}_k^{(b,-1)} = \lim_{s \rightarrow 0} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(b + n) n^{k+s}}{\Gamma(b) n!} \quad (k \neq 0). \quad (59)$$

The polynomials $\mathcal{Q}_k^{(j)}(x)$ are defined as

$$\mathcal{Q}_k^{(j)}(x) \equiv (-i)^k \mathcal{P}_k^{(j)}[ix + (2j - 1)/2], \quad (60)$$

which are listed in (43)-(47), satisfy the three-term recursion relation $\mathcal{Q}_{k+1}^{(j)}(x) = x \mathcal{Q}_k^{(j)}(x) - \frac{k}{4}(k + 2j) \mathcal{Q}_{k-1}^{(j)}(x)$,

and are connected with the Meixner-Pollaczek polynomials $P_k^\lambda(x; \phi)$ [12]

$$\mathcal{Q}_k^{(j)}(x) = 2^{-k} k! P_k^{(j+1/2)}(x; \pi/2). \quad (61)$$

The Meixner-Pollaczek polynomials are defined as

$$P_k(x, \phi) \equiv \frac{\Gamma(2\lambda + k) e^{ik\phi}}{\Gamma(2\lambda) k!} {}_2F_1(-k, ix + \lambda; 2\lambda; 1 - e^{-2i\phi}). \quad (62)$$

For $0 < \phi < \pi$ and $\lambda > 0$, $P_k(x, \phi)$ are continuous orthogonal polynomials on the real line with respect to a continuous positive real measure given by the weight function

$$w(x) = \frac{1}{2\pi} (2 \sin \phi)^{2\lambda} |\Gamma(\lambda + ix)|^2 e^{x(2\phi - \pi)}. \quad (63)$$

The Meixner-Pollaczek polynomials $P_k^\lambda(x)$ are analytic continuations of the Meixner polynomials $\mathcal{M}_n(x, b, c)$ in the parameter c [13]. The connection is given by

$$P_k^\lambda(x) = e^{-ik\phi} \mathcal{M}_k(ix - \lambda, 2\lambda, e^{-2i\phi})/k!, \quad (64)$$

which is like the relation between $\mathcal{Q}_k^{(j)}(x)$ and $\mathcal{P}_k^{(j)}(x)$.

B. Evaluation of moments associated with $\mathcal{Q}_k^{(j)}(x)$

Equations (61) and (64) show the relationship between $\mathcal{Q}_k^{(j)}(x)$ and $\mathcal{M}_k(x)$:

$$\mathcal{Q}_k^{(j)}(x) = (-i)^k 2^{-k} \mathcal{M}_k(ix - j - 1/2, 2j + 1, -1). \quad (65)$$

Moments of the polynomials $\mathcal{Q}_k^{(j)}(x)$ can be evaluated in two ways. One way is to use (64) to express the moments $\mu_k^{(j)}$ associated with the polynomials $\mathcal{Q}_k^{(j)}(x)$ in terms of the regularized moments in (59) associated with the Meixner polynomials

$$\begin{aligned} \mu_k^{(j)} &= (-i)^k \sum_{\alpha=0}^k \binom{k}{\alpha} (j + 1/2)^\alpha \\ &\times \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2j + n + 1)}{n! \Gamma(2j + 1)} n^{k-\alpha}. \end{aligned} \quad (66)$$

For $(2j+1) > 0$ we can express the gamma function in (66) as a sum of powers of n ,

$$\frac{\Gamma(2j+1+n)}{n!} = \sum_{\beta=0}^{2j} (-1)^{2j+\beta} S_{2j}^{(\beta)} \sum_{\sigma=0}^{\beta} \binom{\beta}{\sigma} n^{\sigma}, \quad (67)$$

$$\begin{aligned} \mu_k^{(j)} &= (-1)^k \left\{ (j+1/2)^{2k} + \frac{2^{2j+1}}{\Gamma(2j+1)} \sum_{\alpha=0}^{2k-1} \binom{2k}{\alpha} (j+1/2)^{\alpha} \right. \\ &\quad \times \sum_{\beta=0}^{2j} (-1)^{2j+\beta} S_{2j,\beta}^1 \sum_{\sigma=0}^{\beta} \binom{\beta}{\sigma} (2^{2k+\sigma-\alpha+1} - 1) \zeta(-2k+\alpha-\sigma) \left. \right\}. \end{aligned} \quad (68)$$

Equation (68) reproduces the particular values of $\mathcal{R}_k(j) = 4^k \mu_k^{(j)} / (2j+1)$ given in this paper for $j = 0$ and $j = 1/2$. The special case $\mathcal{R}_k(-1/2)$ is obtained by first dividing (68) by $(2j+2)(2j+1)$ and then substituting the value $j = -1/2$ in the result (53).

For $(2j+1) < 0$ the series over n in (66) becomes finite and regularization is not necessary. The result is

$$\begin{aligned} \mu_k^{(j)} &= (-1)^k 2^{2j+1} \sum_{\alpha=0}^{2k} \binom{2k}{\alpha} (j+1/2)^{\alpha} \\ &\quad \times \Gamma(-2j) \sum_{n=0}^{-2j-1} \frac{n^{2k-\alpha}}{\Gamma(-2j-n) n!}, \end{aligned} \quad (69)$$

which gives the same values for $\mathcal{R}_n(-1)$, $\mathcal{R}_n(-5/2)$ derived earlier in this paper.

The other way is to evaluate the moments of the polynomials $\mathcal{Q}_k^{(j)}(x)$ using their relation with the Meixner-Pollaczek polynomials (61). Using the result [14]

$$\int_{-\infty}^{\infty} dx e^{-(\pi-2\phi)x} |\Gamma(ix+a)|^2 = \pi \Gamma(2a) (2 \sin \phi)^{-2a}, \quad (70)$$

differentiating the above formula n times, and dividing both sides by 2^n , we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dx x^n e^{-(\pi-2\phi)x} |\Gamma(ix+a)|^2 \\ &= 2^{-n} \pi \Gamma(2a) \frac{d^n}{d\phi^n} (2 \sin \phi)^{-2a}. \end{aligned} \quad (71)$$

From (71) with $\phi = \pi/2$ and $a = j+1/2$ and using (61) and (63) we obtain the moments of $\mathcal{Q}_k^{(j)}(x)$:

$$\mu_{2n} = 2^{-2n-1} \frac{d^n}{d\phi^n} (2 \sin \phi)^{-2j-1} \Big|_{\phi=\pi/2}. \quad (72)$$

This result is equivalent to (68) and (69).

Favard's theorem [15] states that if a sequence of monic polynomials satisfies a three-term recursion relation of

and then the zeta regularization (59) gives

the form $xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x)$ with $a_n, b_n \in \mathbb{R}$ and $b_n > 0$, then there exists a positive measure respect to which the polynomials are orthogonal on \mathbb{R} . In our case, because the recursion relation for the polynomials $\mathcal{Q}_n^{(j)}(x)$ is $x\mathcal{Q}_n^{(j)}(x) = \mathcal{Q}_{n+1}^{(j)}(x) + n(n+2j)\mathcal{Q}_{n-1}^{(j)}(x)$, they are orthogonal with respect to a positive measure for $j \geq -1/2$, in agreement with the results in (52) – (54). In Ref. [16] an extension of the Meixner-Pollaczek polynomials for $\lambda \leq 0$ is studied and a nonstandard inner product is defined with respect to which they are orthogonal.

V. CONCLUSIONS

In this paper we have investigated the question of whether the minimal solution for the time operator can be significantly simplified by adding homogeneous solutions. To answer this question we had to make sense of and evaluate infinite sums of singular operators, and to do this we calculated matrix elements of these sums and then used zeta summation to sum the series. The matrix elements give rise to remarkable Meixner polynomials having interesting mathematical properties in which Euler and Bernoulli numbers repeatedly appear.

Our principal conclusion is that the matrix elements of the minimal solution $\langle n-2j | \Theta_{\min} | n \rangle$ for $j = 1, 2, \dots$ and $n \geq 2j$ *cannot all be canceled* by adding one or more homogeneous solutions $\Theta_{\text{hom}}^{(\gamma)}$ for $\gamma = 0, 1, 2, \dots$ to the minimal solution. Specifically, for the minimal solution the matrix elements $\langle n-4j+2 | \Theta_{\min} | n \rangle$ for $j = 1, 2, \dots$ and $n \geq 4j-2$ *can be canceled* by adding to Θ_{\min} the homogeneous solution $\Theta_{\text{hom}}^{(0)}$ with the choice of the parameter $a_0^{(0)} = 1/2$. However, the homogeneous solutions $\Theta_{\text{hom}}^{(\gamma)}$ for $\gamma \geq 1$ cannot cancel *all* the matrix elements of the minimal solution $\langle n-4j+2 | \Theta_{\min} | n \rangle$ for $j = 1, 2, \dots$ and $n \geq 4j-2$.

To elaborate, we have shown that for a finite value of γ , the number of nonzero columns of the operator $\Theta_{\text{hom}}^{(\gamma)}$

is less than the number of the nonzero columns of Θ_{\min} . This means that it is impossible to cancel all the matrix elements $\langle n - 4j + 2 | \Theta_{\min} | n \rangle$. As $\gamma \rightarrow \infty$, the number of nonzero elements in the matrix representation of $\Theta_{\text{hom}}^{(\gamma)}$ becomes equal to the number of nonzero elements in the matrix representation of Θ_{\min} . This means that we can then consider the operator

$$\Theta = \Theta_{\min} + \sum_{\gamma=0}^{\infty} \Theta_{\text{hom}}^{(\gamma)}. \quad (73)$$

The matrix elements in the first nonzero columns above and below the principal diagonals ($j = 1$) are

$$\begin{aligned} \langle n - 2 | \Theta | n \rangle &= -\frac{i}{2} \sqrt{\frac{n}{n-1}} (1 - 2a_0^{(0)}) \\ &\quad - \frac{i}{2} \sqrt{n(n-1)} \sum_{\sigma=0}^{\infty} a_0^{(\sigma+1)} \frac{(2\sigma+2)!}{(\sigma+2)! 2^\sigma \sigma!} \mathcal{P}_\sigma^{(1)}(n). \end{aligned} \quad (74)$$

If we choose

$$a_0^{(\sigma)} = \frac{(\sigma+1)!}{(2\sigma)!} (2t)^{\sigma-1}, \quad (75)$$

with t to be determined, the matrix elements become

$$\begin{aligned} \langle n - 2 | \Theta | n \rangle &= -\frac{(t-1)i}{2t} \sqrt{\frac{n}{n-1}} \\ &\quad + 4i \sqrt{n(n-1)} \frac{(t+2)^{n-2}}{(t-2)^{n+1}}, \end{aligned} \quad (76)$$

where we have used the generating function of the polynomials $\mathcal{P}_\sigma^{(1)}(n)$ to sum the series in (74). The values of t

that make the matrix elements (76) vanish are the roots of the equation

$$t(t-2)^{n+1} - 8(n-1)t - (t-2)^{n+1} = 0 \quad (n \geq 2), \quad (77)$$

with $t \neq 0, 2$.

This demonstrates that to cancel all the elements in a fixed diagonal in the matrix representation $\langle n - 4j + 2 | \Theta | n \rangle$, the choice of the parameter $a_0^{(\sigma)}$ must be different for each n . In general, the value of $a_0^{(\sigma)}$ that cancels the matrix elements $\langle n - 4j + 2 | \Theta | n \rangle$ depends on both j and n .

Acknowledgments

CMB thanks the U.S. Department of Energy for financial support. MG thanks G. Landolfi for useful discussions, the Physics Department of Washington University in St. Louis for its hospitality during the preparation of this work, and the University of Salento and INFN (Lecce) for financial support.

Appendix A: Calculation of Matrix Elements

In this Appendix we show how to calculate the matrix elements in (31). Let us consider the operators

$$I_k = \left(\frac{1}{q} \frac{d}{dq} \right)^k, \quad J_k = \left(\frac{d}{dq} \frac{1}{q} \right)^k \quad (k = 1, 2, \dots). \quad (A1)$$

Here, I_k acts on even eigenstates according to

$$I_k \psi_n(q) = \sum_{\alpha=1}^k \sum_{\beta=0}^{k-\alpha} \frac{\alpha}{\alpha+2\beta} \binom{k}{\alpha+\beta} R_{n,\alpha,\beta} q^{-2\beta-\alpha} H_{n-\alpha}(q) e^{-x^2/2} - H_n(q) e^{-x^2/2}, \quad (A2)$$

where H_n are the Hermite polynomials. Similarly, J_k acts on odd eigenstates and gives the result

$$J_k \psi_n(q) = \sum_{\alpha=0}^k \sum_{\beta=0}^{k-\alpha} \binom{k}{\alpha+\beta} R_{n,\alpha,\beta} q^{-2\beta-\alpha} H_{n-\alpha} e^{-x^2/2}. \quad (A3)$$

The numerical coefficients $R_{n,\alpha,\beta}$ are given by

$$R_{n,\alpha,\beta} = \frac{(-1)^{\alpha+1} 2^{\alpha-\beta} (\alpha+2\beta)! n!}{\pi^{1/4} \sqrt{2^n n!} \alpha! \beta! (n-\alpha)!}. \quad (A4)$$

The Hermite polynomials have the explicit form

$$\begin{aligned} H_{n-\alpha}(q) &= \sum_{l=0}^{\lfloor (n-\alpha)/2 \rfloor} q^{n-\alpha-2l} C_{n,\alpha,l}, \\ C_{n,\alpha,l} &= (-1)^l 2^{n-\alpha-2l} \frac{(n-\alpha)!}{l! (n-\alpha-2l)!}. \end{aligned} \quad (A5)$$

Thus, we can evaluate the integral

$$S_{n,m,\nu} = \int_0^\infty dq e^{-q^2} q^{n-2\nu} H_m(q) \\ = \begin{cases} \frac{2^{m-1} (n/2-\nu)! \Gamma(n/2-\nu+1/2)}{(n/2-\nu-m/2)!} & (m \leq n-2\nu), \\ 0 & \text{otherwise.} \end{cases}$$

The matrix elements $\langle m|I_k|n\rangle$ and $\langle m|J_k|n\rangle$ can then be written in the form

$$\langle m|I_k|n\rangle = \sum_{\alpha=1}^{\alpha_{\max}} \sum_{\beta=0}^{\beta_{\max}} \sum_{\ell=0}^{\ell_{\max}} \frac{\alpha}{\alpha+2\beta} \binom{2k+1}{\alpha+\beta} U_{n,m,\alpha,\ell,\beta}, \\ \langle m|J_k|n\rangle = \sum_{\alpha=0}^{\alpha_{\max}} \sum_{\beta=0}^{\beta_{\max}} \sum_{\ell=0}^{\ell_{\max}} \binom{2k+1}{\alpha+\beta} U_{n,m,\alpha,\ell,\beta}, \quad (\text{A6})$$

where $\alpha_{\max} = (n-m)/2$, $\beta_{\max} = (n-m)/2 - \alpha$, $\ell_{\max} = (n-m)/2 - \alpha - \beta$, and

$$U_{n,m,\alpha,\ell,\beta} = \frac{1}{\pi^{1/4} \sqrt{2^m m!}} R_{n,\alpha,\beta} S_{m,n,\alpha+\ell+\beta} C_{n,\alpha,\ell}. \quad (\text{A7})$$

The coefficients U have the form

$$U_{2n+1,2m+1,\alpha,\ell,\beta} = (-1)^{\alpha+\ell+1} 2^{2\alpha+\beta+m-n} \sqrt{\frac{(2n+1)!}{(2m+1)!}} \frac{(\alpha+2\beta)!(2n-2\alpha-2\beta-2\ell+1)!}{\beta!\ell!\alpha!(n-m-\alpha-\beta-\ell)!(2n-\alpha-2\ell+1)!}, \\ U_{2n,2m,\alpha,\ell,\beta} = (-1)^{\alpha+\ell+1} 2^{2\alpha+\beta+m-n} \sqrt{\frac{(2n)!}{(2m)!}} \frac{(\alpha+2\beta)!(2n-2\alpha-2\beta-2\ell)!}{\beta!\ell!\alpha!(n-m-\alpha-\beta-\ell)!(2n-\alpha-2\ell)!}.$$

With the change of variables $n = 2n'$ and $m = 2n' - 2j$ for $j = 1, 2, \dots, n'$, we make the replacement

$$\tilde{U}_{n,j,\alpha,\ell,\beta} = U_{2n',2n'-2j,\alpha,\ell,\beta}, \quad (\text{A8})$$

and (A6) becomes

$$\langle m|I_k|n\rangle = \sum_{\alpha=1}^j \sum_{\beta=0}^{j-\alpha} \sum_{\ell=0}^{j-\alpha-\beta} \frac{\alpha}{\alpha+2\beta} \binom{2k+1}{\alpha+\beta} \tilde{U}_{n,j,\alpha,\ell,\beta}, \\ \langle m|J_k|n\rangle = \sum_{\alpha=0}^j \sum_{\beta=0}^{j-\alpha} \sum_{\ell=0}^{j-\alpha-\beta} \binom{2k+1}{\alpha+\beta} \tilde{U}_{n,j,\alpha,\ell,\beta}. \quad (\text{A9})$$

Then after the second change of variables, $\alpha = \rho - \sigma$, $\beta = \sigma$, $\tilde{U}_{n,j,\rho,\ell,\sigma} = \tilde{U}_{n,j,\rho-\sigma,\ell,\sigma}$, we obtain

$$\langle m|I_k|n\rangle = \sum_{\rho=1}^j \sum_{\sigma=0}^{\rho-1} \sum_{\ell=0}^{j-\rho} \frac{\rho-\sigma}{\rho+\sigma} \binom{2k+1}{\rho} \tilde{U}_{n,j,\rho,\ell,\sigma}, \\ \langle m|J_k|n\rangle = \sum_{\rho=1}^j \sum_{\sigma=0}^{\rho-1} \sum_{\ell=0}^{j-\rho} \binom{2k+1}{\rho} \tilde{U}_{n,j,\rho,\ell,\sigma}. \quad (\text{A10})$$

Next, we use the formulas

$$\langle 2n-2j|\Theta_{\min}|2n\rangle = -\frac{i}{2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \langle 2n-2j|I_{2k+1}|2n\rangle, \quad (\text{A11})$$

$$\langle 2n+1-2j|\Theta_{\min}|2n+1\rangle = -\frac{i}{2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \langle 2n+1-2j|J_{2k+1}|2n+1\rangle, \quad (\text{A12})$$

to rewrite the even-even and odd-odd matrix elements

$$\langle 2n-2j|\Theta_{\min}|2n\rangle = \mathcal{R}_{n,j}^e \sum_{\rho=1}^j \frac{2^\rho}{(\rho-1)!(j-\rho)!} \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{\delta}, \\ \langle 2n+1-2j|\Theta_{\min}|2n+1\rangle = \mathcal{R}_{n,j}^o \sum_{\delta=1}^j \frac{2^\delta}{(\delta-1)!(j-\delta)!} \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{\rho}, \quad (\text{A13})$$

where for $n = 1, 2, \dots$ and $j = 1, 2, \dots, n$

$$\mathcal{R}_{n,j}^e = i(-1)^j 2^{j/2-1} (j-1)! \sqrt{\frac{n!(2n-2j-1)!!}{(n-j)!(2n-1)!!}}, \\ \mathcal{R}_{n,j}^o = i(-1)^{j+1} 2^{j/2-1} (j-1)! \sqrt{\frac{n!(2n-2j+1)!!}{(n-j)!(2n+1)!!}}.$$

Now an apparent problem arises: The series over k in (A13) is divergent. To overcome this problem we resort to zeta summation to evaluate the divergent series. Zeta summation is conventionally used to evaluate a divergent sum over modes. (For example, it is used in the calculation of the Casimir force [8].) The idea behind zeta summation is to extend $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, which is valid

for $\text{Re } s > 1$, to negative values of s . To do so, one must use the functional equation for the Riemann zeta function

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \quad (\text{A14})$$

We can thus express the divergent series $\sum_{k=1}^{\infty} a(k)$ as $\lim_{s \rightarrow 0} \sum_{k=1}^{\infty} a(k)k^{-s}$. This gives results such as

$$\zeta(0) = \sum_{k=1}^{\infty} k^0 = -\frac{1}{2}, \quad \zeta(-1) = \sum_{k=1}^{\infty} k = -\frac{1}{12}.$$

In our case the sum

$$Z(\rho) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{\rho} \quad (\text{A15})$$

becomes

$$Z(\rho) = \delta_{\rho,1} + \lim_{s \rightarrow 0} \sum_{k=1}^{\infty} \binom{2k+1}{\rho} \frac{k^{-s}}{2k+1}. \quad (\text{A16})$$

It is now straightforward to evaluate the sum over k for generic values of the parameter ρ . We first rewrite the sum by expressing the binomial coefficient as a polynomial in the variable k

$$\binom{2k+1}{\rho} = \sum_{i=1}^{\rho} \frac{S_{\rho,i}^1}{\rho!} (2k+1)^i \quad (\rho = 1, 2, 3 \dots), \quad (\text{A17})$$

where $S_{\rho,i}^1$ are the Stirling numbers of first kind. Then,

$$Z(\rho) = \delta_{\rho,1} + \sum_{i=1}^{\rho} \frac{S_{\rho,i}^1}{\rho!} \sum_{r=0}^{i-1} \binom{i-1}{r} 2^r \zeta(-r). \quad (\text{A18})$$

Because $\zeta(0) = -1/2$, $\zeta(-2n) = 0$ and $\zeta(1-2n) = -\mathcal{B}_{2n}/(2n)$ for $n = 1, 2, 3 \dots$, (A16) is nonzero only for odd values of r and for $r = 0$. Replacing $r \rightarrow 2r-1$, we obtain the final results in (32) and (33).

-
- [1] E. A. Galapon, Proc. R. Soc. Lond. A **458**, 451 (2002).
 - [2] E. A. Galapon, R. F. Caballar, and R. Bahague, Phys. Rev. Lett. **93**, 180406 (2004) and Phys. Rev. A **72**, 062107 (2005).
 - [3] C. M. Bender and G. V. Dunne, Phys. Rev. D **40**, 2739 (1989).
 - [4] C. M. Bender and G. V. Dunne, Phys. Rev. D **40**, 3504 (1989).
 - [5] C. M. Bender and S. P. Klevansky, Phys. Lett. A **373**, 2670 (2009).
 - [6] Another simple relation is $T_{n,n} = P_n(T_{1,1})$, where P_n is the n th Hahn polynomial; this was derived in C. M. Bender, L. R. Mead, and S. S. Pinsky, J. Math. Phys. **28**, 509 (1987).
 - [7] C. M. Bender, Rept. Prog. Phys. **70**, 947-1018 (2007).
 - [8] K. A. Milton, J. Phys. A: Math. Gen. **37**, R209 (2004).
 - [9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (Krieger, New York, 1981), Vol. II, pp. 224-225, 1981.
 - [10] E. W. Weisstein, "Meixner Polynomial of the First Kind." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/MeixnerPolynomialoftheFirstKind.html>
 - [11] C. M. Bender and G. V. Dunne, J. Math. Phys. **xx**, 3504 (1988).
 - [12] R. Koekoek, R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology and Systems, Department of Technical Mathematics and Informatics, Report No. 98-17, (1998).
 - [13] N. M. Atakishiyev and S. K. Suslov, J. Phys. A: Math. Gen. **18**, 1583 (1985).
 - [14] F. Pollaczek, C. R. Acad. Sci. **230**, 1563 (1950).
 - [15] J. Favard, C. R. Acad. Sci. Paris **200**, 2052 (1935).
 - [16] T. K. Araaya, J. Math. Anal. Appl. **305**, 411 (2005).